The concentration-compactness/rigidity method for critical dispersive and wave equations

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In these lectures I will describe a program (which I will call the concentration-compactness/rigidity method) that Frank Merle and I have been developing to study critical evolution problems. The issues studied center around global well-posedness and scattering. The method applies to non-linear dispersive and wave equations in both defocusing and focusing cases. The method can be divided into two parts. The first part ("the concentration-compactness" part) is in some sense "universal" and works in similar ways for "all" critical problems. The second part ("the rigidity" part) has a "universal" formulation, but needs to be established individually for each problem. The method is inspired by the elliptic work on the Yamabe problem and by works of Merle, Martel–Merle and Merle–Raphäel in the non-linear Schrödinger equation and generalized KdV equations.

To focus on the issues, let us first concentrate on the energy critical non-linear Schrödinger equation (NLS) and the energy critical non-linear wave equation (NLW). We thus have:

$$\begin{cases} i\partial_t u + \triangle u \pm |u|^{4/N-2} u = 0 & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u\big|_{t=0} = u_0 \in \dot{\mathrm{H}}^1(\mathbb{R}^n) & N \geq 3 \end{cases} \tag{1}$$

and

$$\begin{cases} \partial_t^2 u - \triangle u = \pm |u|^{4/N-2} u & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u\big|_{t=0} = u_0 \in \dot{\mathrm{H}}^1(\mathbb{R}^n) & \\ \partial_t u\big|_{t=0} = u_1 \in L^2(\mathbb{R}^n) & N \geq 3 \end{cases} \tag{2}$$

In both cases, the "$-$" sign corresponds to the defocusing case, while the "$+$" sign corresponds to the focusing case. For (1), if u is a solution, so is $\frac{1}{\lambda^{N-2/2}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$. For (2), if u is a solution, so is $\frac{1}{\lambda^{N-2/2}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right)$. Both scalings leave invariant the energy spaces $\dot{\mathrm{H}}^1$, $\dot{\mathrm{H}}^1 \times L^2$ respectively and that is why they are called energy critical. The energy which is conserved in this problem is

$$E_\pm(u_0) = \frac{1}{2}\int |\nabla u_0|^2 \pm \frac{1}{2^*}\int |u_0|^{2^*}, \qquad (NLS)$$

$$E_\pm((u_0,u_1)) = \frac{1}{2}\int |\nabla u_0|^2 + \frac{1}{2}\int |u_1|^2 \pm \frac{1}{2^*}\int |u_0|^{2^*}, \qquad (NLW)$$

where $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N} = \frac{N-2}{2N}$. The "$+$" corresponds to the defocusing case while the "$-$" corresponds to the focusing case.

In both problems, the theory of the local Cauchy problem has been understood for a while (in the case of (1) , through the work of Cazenave–Weissler [7], while in the case of (2) through the works of Pecher [37], Ginibre–Velo [14], Ginibre-Velo-Soffer [13] and many others, for instance [20], [34], [41], [3] etc.). These works show that, say for (1), for any u_0, $\|u_0\|_{\dot{\mathrm{H}}^1} \leq \delta$, there exists a unique solution of (1) defined for all time and the solution scatters, i.e. there exist u_0^+, u_0^- in $\dot{\mathrm{H}}^1$ such that

$$\lim_{t\to\pm\infty} \left\|u(t) - \mathrm{e}^{it\Delta} u_0^\pm\right\|_{\dot{\mathrm{H}}^1} = 0.$$

A corresponding result holds for (2). Moreover, given any initial data u_0 ((u_0, u_1)) in the energy space, there exist $T_+(u_0)$, $T_-(u_0)$ such that there exists a unique solution in $(-T_-(u_0), T_+(u_0))$ and the interval is maximal (for (2) $(-T_-(u_0, u_1)$, $T_+(u_0, u_1))$). In both problems, there exists a crucial space-time norm (or "Strichartz norm"). For (1), on a time interval I, we define

$$||u||_{S(I)} = ||u||_{L_I^{2(N+2)/N-2} L_x^{2(N+2)/N-2}},$$

while for (2) we have

$$||u||_{S(I)} = ||u||_{L_I^{2(N+1)/N-2} L_x^{2(N+1)/N-2}}.$$

This norm is crucial, say for (1) because, if $T_+(u_0) < +\infty$, we must have $||u||_{S((0,T_+(u_0)))} = +\infty$, moreover, if $T_+(u_0) = +\infty$, u scatters at $+\infty$ if and only if $||u||_{S(0,+\infty)} < +\infty$. Similar results hold for (2). The question that attracted people's attention here is: What happens for large data? The question was first studied for (2) in the defocusing case, through works of Struwe ([44]) in the radial case, Grillakis ([16], [17]) in the general case, for the preservation of smoothness, and in the terms described here in the works of Shatah–Struwe [41], [42], Bahouri–Shatah [3], Bahouri–Gérard [2], Kapitansky [20], etc. The summary of these works is that (this was achieved in the early 90's), for any pair $(u_0, u_1) \in \dot{\mathrm{H}}^1 \times L^2$, in the defocusing case we have $T_\pm(u_0, u_1) = +\infty$ and the solution scatters. The corresponding results for (1) in the defocusing case took much longer. The first result was established by Bourgain [4], in 1998, who established the analogous result for u_0 radial, $N = 3, 4$, with Grillakis [18] showing preservation of smoothness for $N = 3$, radial data. Tao extended these results to $N \geq 5$, u_0 radial [48]. Finally, Colliander–Kell–Staffilani–Takaoka–Tao proved this for $N = 3$, all data u_0 [8], with extensions to $N = 4$ by Ryckman-Vişan [40] and to $N \geq 5$ by Vişan [54] in 2005.

In the focusing case, these results do not hold. In fact, for (2) H. Levine [33] in 1974 showed that in the focusing case, if $(u_0, u_1) \in \dot{\mathrm{H}}^1 \times L^2$, $u_0 \in L^2$ and $E((u_0, u_1)) < 0$, there is always a break-down in finite time i.e. $T_\pm(u_0, u_1) < \infty$. He showed this by an "obstruction" type of argument. Recently Krieger–Schlag–Tătaru [32] have constructed radial examples ($N = 3$), for which $T_\pm(u_0, u_1) < \infty$. For (1) a classical argument due to Zakharov and Glassey [15], based on the virial identity, shows the same result as H. Levine's if $\int |x|^2 |u_0|^2 < \infty$, $E(u_0) < 0$. Moreover, for both (1) and (2), in the focusing case we have the following static solution

$$W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-(N-2)/2} \in \dot{\mathrm{H}}^1(\mathbb{R}^N),$$

which solves the elliptic equation

$$\triangle W + |W|^{4/N-2}W = 0.$$

Thus, scattering need not occur for solutions that exist globally in time. The solution W plays an important role in the Yamabe problem (see [1] for instance) and it does so once more here. The results in which I am going to concentrate here are:

Theorem 1 (Kenig–Merle [25]). *For the focusing energy critical (NLS),* $3 \le N \le 6$, *consider* $u_0 \in \dot{\mathrm{H}}^1$ *such that* $E(u_0) < E(W)$, u_0 *radial. Then:*

i) If $\|u_0\|_{\dot{\mathrm{H}}^1} < \|W\|_{\dot{\mathrm{H}}^1}$, *the solution exists for all time and scatters.*

ii) If $||u_0||_{L^2} < \infty$, $\|u_0\|_{\dot{\mathrm{H}}^1} > \|W\|_{\dot{\mathrm{H}}^1}$, *then* $T_+(u_0) < +\infty$, $T_-(u_0) < +\infty$.

Remark 1. Recently, Killip–Vişan [29] have combined the ideas of the proof of Threorem 2, as applied to NLS in [10], with another important new idea, to extend Theorem 1 to the non-radial case for $N \ge 5$.

The one case where we don't need the radial assumption in dimensions $3 \le N \le 6$ is the one of (2). We have:

Theorem 2 (Kenig–Merle [23]). *For the focusing energy critical (NLW),* $3 \le N \le 6$, *consider* $(u_0, u_1) \in \dot{\mathrm{H}}^1 \times L^2$ *such that* $E((u_0, u_1)) < E((W, 0))$. *Then*

i) If $\|u_0\|_{\dot{\mathrm{H}}^1} < \|W\|_{\dot{\mathrm{H}}^1}$, *the solution exists for all time and scatters.*

ii) If $\|u_0\|_{\dot{\mathrm{H}}^1} > \|W\|_{\dot{\mathrm{H}}^1}$, *then* $T_\pm(u_0) < +\infty$.

I will sketch the proofs of these two theorems and the outline of the general method in these lectures. The method has found other interesting applications:

Mass Critical NLS

$$\begin{cases} i\partial_t u + \triangle u \pm |u|^{4/N}u = 0 & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u\big|_{t=0} = u_0 & N \ge 3 \end{cases} \tag{3}$$

Here, $||u_0||_{L^2}$ is the critical norm. The analog of Theorem 1 was obtained, for u_0 radial by Tao–Vişan–Zhang [50], Killip–Tao–Vişan [28], Killip–Vişan-Zhang [30], using our proof scheme for $N \ge 2$. (In the focusing case one needs to assume $||u_0||_{L^2} < ||Q||_{L^2}$, where Q is the ground state, i.e. the non-negative solution of the elliptic equation $\triangle Q + Q^{1+4/N} = Q$). The case $N = 1$ is open.

Corotational wave maps into S^2, 4D Yang–Mills in the radial case Consider the wave map system

$$\Box u = A(u)(Du, Du) \perp T_u M$$

where $u = (u^1, \dots, u^d) : \mathbb{R} \times \mathbb{R}^N \to M \hookrightarrow \mathbb{R}^d$, where M, the target manifold is isometrically embedded in $\mathbb{R}^d$, and $A(u)$ is the second fundamental form for M at u. We consider the case $M = S^2 \subset \mathbb{R}^3$. The critical space here is $(u_0, u_1) \in \dot{H}^{N/2} \times \dot{H}^{N-2/2}$, so that when $N = 2$, the critical space is $\dot{\mathrm{H}}^1 \times L^2$. It is known that for small data in $\dot{\mathrm{H}}^1 \times L^2$ we have global existence and scattering (Tătaru [52], [53], Tao [47]). Moreover, Rodnianski–Sterbenz [39] and Krieger–Schlag–Tătaru [31], showed that there can be finite time blow-up for large data. In earlier work, Struwe [45] had considered the case of co-rotational maps. These are maps which have a special form. Writing the metric on S^2 in the form (ρ, θ), $\rho > 0$, $\theta \in S^1$, with $ds^2 = d\rho^2 + g(\rho)^2 d\theta^2$, where $g(\rho) = \sin \rho$, we consider, using (r, ϕ) as polar coordinates in $\mathbb{R}^2$, maps of the form $\rho = v(r, t)$, $\theta = \phi$. These are the co-rotational maps and Krieger–Schlag–Tătaru [31] exhibited blow-up for corotational maps. There is a stationary solution Q, which is a non-constant harmonic map of least energy. Struwe proved that if $E(v) \leq E(Q)$, v and the corresponding wave map u are global in time. Using our method, in joint of Cote–Kenig–Merle [9] we show that, in addition, there is an alternative: $v \equiv Q$ or the solution scatters. We also prove the corresponding results for radial solutions of the Yang–Mills equations in the critical energy space in $\mathbb{R}^4$ (see [9]).

Cubic NLS in 3d: Consider the classic cubic NLS in 3d:

$$\begin{cases} i\partial_t u + \triangle u \mp |u|^2 u = 0 & (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u\big|_{t=0} = u_0 \in \dot{H}^{1/2}(\mathbb{R}^3) \end{cases}$$

$\dot{H}^{1/2}$ is the critical space, "$-$" = defocusing, "$+$" = focusing. In the focusing case, Duyckaerts-Holmer-Roudenko [10] adapted our method to show that if $u_0 \in \dot{\mathrm{H}}^1(\mathbb{R}^3)$ and $M(u_0)E(u_0) < M(Q)E(Q)$, where

$$M(u_0) = \int |u_0|^2, \qquad E(u_0) = \frac{1}{2}\int |\nabla u_0|^2 - \frac{1}{4}\int |u_0|^4$$

and Q is the ground state, i.e. the positive solution to the elliptic equation

$$-Q + \triangle Q + |Q|^2 Q = 0,$$

then if $||u_0||_{L^2}||\nabla u_0||_{L^2} > ||Q||_{L^2}||\nabla Q||_{L^2}$, we have "blow-up" in finite time, while if $||u_0||_{L^2}||\nabla u_0||_{L^2} < ||Q||_{L^2}||\nabla Q||_{L^2}$, u exists for all time and scatters. In joint work with Merle [24] we have considered the defocusing case. We have shown there, using this circle of ideas that if $\sup_{0<t<T_+(u_0)} ||u(t)||_{\dot{H}^{1/2}} < \infty$, then $T_+(u_0) = +\infty$ and u scatters. We would like to point out that the fact that $T_+(u_0) = +\infty$ is analogous to the $L^{3,\infty}$ result of Escauriaza–Seregin–Sverak for Navier–Stokes [11].

We now turn to the proofs of Theorem 1, 2. We start with Theorem 1. We are thus considering

$$\begin{cases} i\partial_t u + \triangle u + |u|^{4/N-2}u = 0 \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u\big|_{t=0} = u_0 \in \dot{\mathrm{H}}^1 \end{cases} \tag{4}$$

Let us start with a quick review of the "local Cauchy problem" theory. Besides the norm $\|f\|_{S(I)} = \|f\|_{L_I^{2(N+2)/N-2} L_x^{2(N+2)/N-2}}$ introduced earlier, we need the norm $\|f\|_{W(I)} = \|f\|_{L_I^{2(N+2)/N-2} L_x^{2(N+2)/N^2+4}}$.

Theorem 3 ([7], [25])**.** *Assume that $u_0 \in \dot{\mathrm{H}}^1(\mathbb{R}^N)$, $\|u_0\|_{\dot{\mathrm{H}}^1} \leq A$. Then (for $3 \leq N \leq 6$) there exists $\delta = \delta(A) > 0$ such that if $\left\|\mathrm{e}^{it\Delta}u_0\right\|_{S(I)} \leq \delta$, $0 \in \mathring{I}$, there exists a unique solution to (4) in $\mathbb{R}^N \times I$, with $u \in C(I; \dot{\mathrm{H}}^1)$ and $\|\nabla u\|_{W(I)} < +\infty$, $\|u\|_{S(I)} \leq 2\delta$. Moreover, the mapping $u_0 \in \dot{\mathrm{H}}^1(\mathbb{R}^N) \to u \in C(I; \dot{\mathrm{H}}^1)$ is Lipschitz.*

The proof is by fixed point. The key ingredients are the following "Strichartz estimates" ([43], [21]):

$$\begin{cases} \left\|\nabla \mathrm{e}^{it\Delta}u_0\right\|_{W(-\infty,+\infty)} \leq C\left\|u_0\right\|_{\dot{\mathrm{H}}^1} \\ \left\|\nabla \int_0^t \mathrm{e}^{i(t-t')\Delta} g(\cdot,t')dt'\right\|_{W(-\infty,+\infty)} \leq C\left\|g\right\|_{L_t^2 L_x^{2N/N+2}} \\ \sup_t \left\|\nabla \int_0^t \mathrm{e}^{i(t-t')\Delta} g(\cdot,t')dt'\right\|_{L^2} \leq C\left\|g\right\|_{L_t^2 L_x^{2N/N+2}} \end{cases} \tag{5}$$

and the following Sobolev embedding

$$\|v\|_{S(I)} \leq C\|\nabla v\|_{W(I)}, \tag{6}$$

and the observation that $\left|\nabla(|u|^{4/N-2}u)\right| \leq C|\nabla u|\ |u|^{4/N-2}$, so that

$$\left\|\nabla(|u|^{4/N-2}u)\right\|_{L_I^2 L_x^{2N/N+2}} \lesssim \|u\|_{S(I)}^{4/N-2} \|\nabla u\|_{W(I)}.$$

Remark 2. Because of (5), (6), there exists $\tilde{\delta}$ such that if $\|u_0\|_{\dot{\mathrm{H}}^1} \leq \tilde{\delta}$, the hypothesis of the Theorem is verified for $I = (-\infty, +\infty)$. Moreover, given $u_0 \in \dot{\mathrm{H}}^1$, we can find I such that $\left\|\mathrm{e}^{it\Delta}u_0\right\|_{S(I)} < \delta$, so that the Theorem applies. It is then easy to see that given $u_0 \in \dot{\mathrm{H}}^1$, there exists a maximal interval $I = (-T_-(u_0), T_+(u_0))$ where $u \in C(I'; \dot{\mathrm{H}}^1) \cap \{\nabla u \in W(I')\}$ of all $I' \subset\subset I$ is defined. We call I the maximal interval of existence. It is easy to see that for all $t \in I$, we have

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t)|^2 - \frac{1}{2^*}\int |u|^{2^*} = E(u_0).$$

We also have the "standard finite time blow-up criterion": if $T_+(u_0) < \infty$, then $||u||_{S([0,T_+(u_0)))} = +\infty$.

We next turn to another fundamental result in the "local Cauchy theory", the so called "Perturbation Theorem".

Perturbation Theorem 4 (see [49], [25], [22]). *Let $I = [0, L)$, $L \le +\infty$, $\tilde{u}$ be defined on $\mathbb{R}^N \times I$ be such that*

$$\sup_{t\in I} \|\tilde{u}\|_{\dot{\mathrm{H}}^1} \le A, \quad \|\tilde{u}\|_{S(I)} \le M, \quad \|\nabla \tilde{u}\|_{W(I)} < +\infty$$

and verify (in the sense of the integral equation)

$$i\partial_t \tilde{u} + \triangle \tilde{u} + |\tilde{u}|^{4/N-2}\tilde{u} = e \quad \text{on } \mathbb{R}^N \times I,$$

and let $u_0 \in \dot{\mathrm{H}}^1$ be such that $\|u_0 - \tilde{u}(0)\|_{\dot{\mathrm{H}}^1} \le A'$. Then, there exists $\epsilon_0 = \epsilon_0(M, A, A')$ such that, if $0 \le \epsilon \le \epsilon_0$ and $\|\nabla e\|_{L^2_I L_x^{2N/N+2}} \le \epsilon$, $\left\|\mathrm{e}^{it\Delta}[u_0 - \tilde{u}(0)]\right\|_{S(I)} \le \epsilon$, then there exists a unique solution u to (4) on $\mathbb{R}^N \times I$, such that

$$\|u\|_{S(I)} \le C(A, A', M) \quad \text{and} \quad \sup_{t\in I} \|u(t) - \tilde{u}(t)\|_{\dot{\mathrm{H}}^1} \le C(A, A', M)(A' + \epsilon)^{\beta},$$

where $\beta > 0$.

For the details of the proof see [22]. This result has several important consequences:

Corollary 1. *Let $K \subset \dot{\mathrm{H}}^1$ be such that $\overline{K}$ is compact. Then, $\exists\ T_{+,\overline{K}}$, $T_{-,\overline{K}}$ such that $\forall u_0 \in K$ we have $T_+(u_0) \ge T_{+,\overline{K}}$, $T_-(u_0) \ge T_{-,\overline{K}}$.*

Corollary 2. *Let $\tilde{u}_0 \in \dot{\mathrm{H}}^1$, $\|\tilde{u}_0\|_{\dot{\mathrm{H}}^1} \le A$, let $\tilde{u}$ be the solution of (4), with maximal interval $(-T_-(\tilde{u}_0), T_+(\tilde{u}_0))$. Assume that $u_{0,n} \to \tilde{u}_0$ in $\dot{\mathrm{H}}^1$, with corresponding solution u_n. Then $T_+(\tilde{u}_0) \le \underline{\lim}\, T_+(u_{0,n})$, $T_-(\tilde{u}_0) \le \underline{\lim}\, T_-(u_{0,n})$ and for $t \in (-T_-(\tilde{u}_0), T_+(\tilde{u}_0))$, $u_n(t) \to \tilde{u}(t)$ in $\dot{\mathrm{H}}^1$.*

Before we start with our sketch of the proof of Theorem 1, we will review the classic argument of Glassey [15] for blow-up in finite time. Thus, assume $u_0 \in \dot{\mathrm{H}}^1$, $\int |x|^2 |u_0(x)|^2 dx < \infty$ and $E(u_0) < 0$. Let I = maximal interval of existence. One easily shows that, for $t \in I$, $y(t) = \int |x|^2 |u(x,t)|^2 dx < +\infty$. In fact,

$$y'(t) = 4\,\mathrm{Im}\!\int \overline{u} \nabla u \cdot x, \quad \text{and} \quad y''(t) = 8\left[\int |\nabla u(x,t)|^2 - \int |u(x,t)|^{2^*}\right].$$

Hence, if $E(u_0) < 0$, $E(u(t)) = E(u_0) < 0$, so that

$$\frac{1}{2}\int |\nabla u(t)|^2 - |u(t)|^{2^*} = E(u_0) + \left(\frac{1}{2^*} - \frac{1}{2}\right)\int |u(t)|^{2^*} \le E(u_0) < 0,$$

and $y''(t) < 0$. But then, if I is infinite, since $y(t) > 0$ we obtain a contradiction.

We now start with our sketch of the proof of Theorem 1.

Step 1: Variational estimates. (These are not needed in defocusing problems). Recall that $W(x) = (1 + |x|^2/N(N-2))^{-(N-2)/2}$ is a stationary solution of (4). It solves the elliptic equation $\triangle W + |W|^{4/N-2}W = 0$, $W \geq 0$, W is radially decreasing, $W \in \dot{\mathrm{H}}^1$. By the invariances of the equation,

$$W_{\theta_0,x_0,\lambda_0}(x) = e^{i\theta_0}\lambda_0^{N-2/2}W(\lambda_0(x - x_0))$$

is still a solution. Aubin and Talenti ([1], [46]) gave the following variational characterization of W: let C_N be the best constant in the Sobolev embedding $||u||_{L^{2^*}} \leq C_N||\nabla u||_{L^2}$. Then, $||u||_{L^{2^*}} = C_N||\nabla u||_{L^2}$, $u \not\equiv 0$, if and only if $u = W_{\theta_0,x_0,\lambda_0}$ for some $(\theta_0, x_0, \lambda_0)$. Note that by the elliptic equation, $\int|\nabla W|^2 = \int|W|^{2^*}$. Also, $C_N||\nabla W|| = ||W||_{L^{2^*}}$, so that

$$C_N^2||\nabla W||^2 = \left(\int|\nabla W|^2\right)^{\frac{N-2}{N}}.$$

Hence, $\int|\nabla W|^2 = 1/C_N^N$, and

$$E(W) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\int|\nabla W|^2 = \frac{1}{NC_N^N}.$$

Lemma 1. *Assume that* $||\nabla v|| < ||\nabla W||$ *and that* $E(v) \leq (1 - \delta_0)E(W)$, $\delta_0 > 0$. *Then, there exists* $\overline{\delta} = \overline{\delta}(\delta_0)$ *so that:*

i) $||\nabla v||^2 \leq (1 - \overline{\delta})||\nabla W||^2$

ii) $\displaystyle\int|\nabla v|^2 - |v|^{2^*} \geq \overline{\delta}||\nabla v||^2$

iii) $E(v) \geq 0$

Proof. Let

$$f(y) = \frac{1}{2}y - \frac{C_N^{2^*}}{2^*}y^{2^*/2}, \quad \overline{y} = ||\nabla v||^2.$$

Note that $f(0) = 0$, $f(y) > 0$ for y near 0, $y > 0$ and that

$$f'(y) = \frac{1}{2} - \frac{C_N^{2^*}}{2^*}y^{2^*/2-1},$$

so that $f'(y) = 0$ if and only if $y = y_c = \frac{1}{C_N} = ||\nabla W||^2$. Also, $f(y_c) = \frac{1}{NC_N} = E(W)$. Since $0 \leq \overline{y} < y_c$, $f(\overline{y}) \leq (1 - \delta_0)f(y_c)$, f is non-negative and strictly increasing between 0 and y_c, and $f''(y_c) \neq 0$, we have $0 \leq f(\overline{y})$, $\overline{y} \leq (1 - \overline{\delta})y_c = (1 - \overline{\delta})||\nabla W||^2$. This shows i).

For ii), note that

$$\begin{aligned}
\int |\nabla v|^2 - |v|^{2^*} &\geq \int |\nabla v|^2 - C_N^{2^*}\left(\int |\nabla v|^2\right)^{2^*/2} \\
&= \int |\nabla v|^2 \left[1 - C_N^{2^*}\left(\int |\nabla v|^2\right)^{2/N-2}\right] \\
&\geq \int |\nabla v|^2 \left[1 - C_N^{2^*}(1-\overline{\delta})^{2/N-2}\left(\int |\nabla W|^2\right)^{2/N-2}\right] \\
&= \int |\nabla v|^2 \left[1 - (1-\overline{\delta})^{2/N-2}\right]
\end{aligned}$$

which gives ii).

Note from this that if $||\nabla u_0|| < ||\nabla W||$, then $E(u_0) \geq 0$, i.e. iii). □

This static lemma imediately has dynamic consequences.

Corollary 3 (Energy Trapping). *Let u be a solution of (4) with maximal interval I, $||\nabla u_0|| < ||\nabla W||$, $E(u_0) < E(W)$. Choose $\delta_0 > 0$ such that $E(u_0) \leq (1-\delta_0)E(W)$. Then, for each $t \in I$ we have:*

i) $||\nabla u(t)||^2 \leq (1-\overline{\delta})||\nabla W||, \quad E(u(t)) \geq 0$

ii) $\int |\nabla u(t)|^2 - |u(t)|^{2^*} \geq \overline{\delta} \int |\nabla u(t)|^2$ *(Coercivity)*

iii) $E(u(t)) \approx ||\nabla u(t)||^2 \approx ||\nabla u_0||^2$*, with comparability constants which depend on δ_0. (Uniform bound).*

Proof. From continuity of the flow, conservation of energy and the previous Lemma. □

Note that iii) gives uniform bounds on $||\nabla u(t)||$. However, this is a long way from giving Theorem 1.

Remark 3. Let $u_0 \in \dot{\mathrm{H}}^1$, $E(u_0) < E(W)$, but $||\nabla u_0||^2 > ||\nabla W||^2$. If we choose δ_0 so that $E(u_0) \leq (1-\delta_0)E(W)$, we can conclude, as in the proof of the Lemma, that $\int |\nabla u(t)|^2 \geq (1+\overline{\delta}) \int |\nabla W|^2$, $t \in I$. But then,

$$\begin{aligned}
\int |\nabla u(t)|^2 - |u(t)|^{2^*} &= 2^* E(u_0) - \frac{2}{N-2}\int |\nabla u|^2 \\
&\leq 2^* E(W) - \frac{2}{N-2}\frac{1}{C_N^N} - \frac{2\overline{\delta}}{N-2}\frac{1}{C_N^N} \\
&= -\frac{2\overline{\delta}}{(N-2)C_N^N} < 0.
\end{aligned}$$

Hence, if $\int |x|^2 |u_0(x)|^2 dx < \infty$, Glassey's proof shows that I cannot be infinite. If u_0 is radial, $u_0 \in L^2$, using a "local virial identity" (which we will see momentarily) one can see that the same result holds.

Step 2: Concentration-compactness procedure. We now turn to the proof of i) in Theorem 1. By our variational estimates, if $E(u_0) < E(W)$, $||\nabla u_0||^2 < ||\nabla W||^2$, if δ_0 is chosen so that $E(u_0) \leq (1-\delta_0)E(W)$, recall that

$$E(u(t)) \approx ||\nabla u(t)||^2 \approx ||\nabla u_0||^2,$$

$t \in I$, with constants depending only on δ_0, recall also that if $||\nabla u_0||^2 < ||\nabla W||^2$, $E(u_0) \geq 0$. It now follows from the "local Cauchy theory" that if $||\nabla u_0||^2 < ||\nabla W||^2$ and $E(u_0) \leq \eta_0$, η_0 small, then $I = (-\infty, +\infty)$ and $||u||_{S(-\infty,+\infty)} < \infty$, so that u scatters. Consider now

$$G = \Big\{ E : 0 < E < E(W) \quad : \quad \text{if } ||\nabla u_0||^2 < ||\nabla W||^2 \quad \text{and } E(u_0) < E, \text{ then } \|u\|_{S(I)} < \infty \Big\}$$

and $E_c = \sup G$. Then, $0 < \eta_0 \leq E_c \leq E(W)$ and if $||\nabla u_0||^2 < ||\nabla W||^2$, $E(u_0) < E_c$, $I = (-\infty, +\infty)$, u scatters and E_c is optimal with this property. Theorem 1 i) is the statement $E_c = E(W)$. We now assume $E_c < E(W)$ and will reach a contradiction. We now develop the concentration-compactness argument:

Proposition 1. *There exists $u_{0,c} \in \dot{\mathrm{H}}^1$, $||\nabla u_{0,c}||^2 < ||\nabla W||^2$, with $E(u_{0,c}) = E_c$, such that, for the corresponding solution u_c , we have $\|u_c\|_{S(I)} = +\infty$.*

Proposition 2. *For any u_c as in Proposition 1, with (say) $||u_c||_{S(I_+)} = +\infty$, $I_+ = I \cap [0, +\infty)$, there exist $x(t)$, $t \in I_+$, $\lambda(t) \in \mathbb{R}^+$, $t \in I_+$, such that*

$$K = \left\{ v(x,t) = \frac{1}{\lambda(t)^{N-2/2}} u\left(\frac{x - x(t)}{\lambda(t)}, t\right), t \in I_+ \right\}$$

has compact closure in $\dot{\mathrm{H}}^1$.

The proof of Proposition 1 and 2 follows a "general procedure" which uses a "profile decomposition", the variational estimates and the "Perturbation Theorem". The idea of the decomposition is somehow a time-dependent version of the concentration-compactness method of P.L. Lions, when the "local Cauchy theory" is done in the critical space. It was introduced independently by Bahouri–Gérard [2] for the wave equation and by Merle–Vega for the L^2 critical NLS [35]. The version needed for Theorem 1 is due to Keraani [27]. This is the evolution analog of the elliptic "bubble decomposition", which goes back to work of Brézis–Coron [5].

Theorem 5 (Keraani [27])**.** *Let $\{v_{0,n}\} \subset \dot{\mathrm{H}}^1$, with $\|v_{0,n}\|_{\dot{\mathrm{H}}^1} \leq A$. Assume that $\left\|\mathrm{e}^{it\Delta} v_{0,n}\right\|_{S(-\infty,+\infty)} \geq \delta > 0$. Then there exists a subsequence of $\{v_{0,n}\}$ and a sequence $\{V_{0,j}\}_{j=1}^{\infty} \subset \dot{\mathrm{H}}^1$ and triplets $\{(\lambda_{j,n}, x_{j,n}, t_{j,n})\} \subset \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}$, with*

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}^2} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \xrightarrow[n\to\infty]{} \infty,$$

for $j \neq j'$ (we say that $\{(\lambda_{j,n}, x_{j,n}, t_{j,n})\}$ is orthogonal), such that

i) $\|V_{0,1}\|_{\dot{\mathrm{H}}^1} \geq \alpha_0(A) > 0$

ii) If $V_j^l(x,t) = \mathrm{e}^{it\Delta} V_{0,j}$, then we have, for each J,

$$v_{0,n} = \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{N-2/2}} V_j^l \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, -\frac{t_{j,n}}{\lambda_{j,n}^2} \right) + w_n^J,$$

where $\underline{\lim}_n \left\| \mathrm{e}^{it\Delta} w_n^J \right\|_{S(-\infty,+\infty)} \xrightarrow[J\to\infty]{} 0$ *and for each $J \geq 1$, we have*

iii) $||\nabla v_{0,n}||^2 = \sum_{j=1}^{J} ||\nabla V_{0,j}||^2 + ||\nabla w_n^J||^2 + o(1)$ *as $n \to \infty$ and*

$$E(v_{0,n}) = \sum_{j=1}^{J} E\left(V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right) \right) + E(w_n^J) + o(1) \text{ as } n \to \infty.$$

Further general remarks:

Remark 4. Because of the continuity of $u(t)$, $t \in I$, in $\dot{\mathrm{H}}^1$, in Proposition 2 we can construct $\lambda(t)$, $x(t)$ continuous in $[0, T_+(u_0))$, with $\lambda(t) > 0$.

Remark 5. Because of scaling and the compactness of $\overline{K}$ above, if $T_+(u_{0,c}) < \infty$, one always has that $\lambda(t) \geq C_0(K)/(T_+(u_0,c) - t)^{\frac{1}{2}}$.

Remark 6. If $T_+(u_{0,c}) = +\infty$, we can always find another (possibly different) critical element v_c, with a corresponding $\tilde{\lambda}$ so that $\tilde{\lambda}(t) \geq A_0 > 0$ for $t \in [0, T_+(v_{0,c}))$. (Again by compactness of $\overline{K}$).

Remark 7. One can use the "profile decomposition" to also show that there exists a decreasing function g, $g : (0, E_c] \to [0, +\infty)$ so that if $||\nabla u_0||^2 < ||\nabla W||^2$ and $E(u_0) \leq E_c - \eta$, then $\|u\|_{S(-\infty,+\infty)} \leq g(\eta)$.

Remark 8. In the "profile decomposition", if all the $v_{0,n}$ are radial, the $V_{0,j}$ can be chosen radial and $x_{j,n} \equiv 0$. We can repeat our procedure restricted to radial data and conclude the analog of Proposition 1, 2, with $x(t) \equiv 0$.

The final step in the proof is then:

Step 3: Rigidity theorem

Theorem 6 (Rigidity)**.** *Let $u_0 \in \dot{\mathrm{H}}^1$, $E(u_0) < E(W)$, $||\nabla u_0||^2 < ||\nabla W||^2$. Let u be the solution of (4), with maximal interval $I = (-T_-(u_0), T_+(u_0))$. Assume that there exists $\lambda(t) > 0$, defined for $t \in [0, T_+(u_0))$, such that*

$$K = \left\{ v(x,t) = \frac{1}{\lambda(t)^{N-2/2}} u\left(\frac{x}{\lambda(t)}, t \right), t \in [0, T_+(u_0)) \right\}$$

has compact closure in $\dot{\mathrm{H}}^1$. *Assume also that if* $T_+(u_0) < \infty$, $\lambda(t) \geq C_0(K)/(T_+(u_0,c) - t)^{\frac{1}{2}}$ *and if* $T_+(u_0) = \infty$, *that* $\lambda(t) \geq A_0 > 0$ *for* $t \in [0, +\infty)$. *Then,* $T_+(u_0) = +\infty$, $u_0 \equiv 0$.

To prove this, we split two cases:

Case 1: $T_+(u_0) < +\infty$ (so that $\lambda(t) \to +\infty$ as $t \to T_+(u_0)$)

Fix ϕ radial, $\phi \in C_0^\infty$, $\phi \equiv 1$ on $|x| \leq 1$, $\operatorname{supp} \phi \subset \{|x| < 2\}$. Set $\phi_R(x) = \phi(x/R)$ and define

$$y_R(t) = \int |u(x,t)|^2 \phi_R(x)\, dx.$$

Then, $y_r'(t) = 2\,\mathrm{Im} \int \overline{u} \nabla u \nabla \phi_R$, so that

$$|y_R'(t)| \leq C \left(\int |\nabla u|^2 \right)^{1/2} \left(\int \frac{|u|^2}{|x|^2} \right)^{1/2} \leq C ||\nabla W||^2,$$

by Hardy's inequality and our variational estimates. Note that C is independent of R. Next, we note that, for each $R > 0$,

$$\lim_{t \uparrow T_+(u_0)} \int_{|x|<R} |u(x,t)|^2 dx = 0.$$

In fact, $u(x,t) = \lambda(t)^{N-2/2} v(\lambda(t)x, t)$, so that

$$\begin{aligned} \int_{|x|<R} |u(x,t)|^2 dx &= \lambda(t)^{-2} \int_{|y|<R\lambda(t)} |v(y,t)|^2 dy \\ &= \lambda(t)^{-2} \int_{|y|<\epsilon R\lambda(t)} |v(y,t)|^2 dy \\ &+ \lambda(t)^{-2} \int_{\epsilon R\lambda(t) \leq |y| < R\lambda(t)} |v(y,t)|^2 dy \\ &= A + B. \end{aligned}$$

$$A \leq \lambda(t)^{-2} (\epsilon R \lambda(t))^2 ||v||_{L^{2^*}}^2 \leq C \epsilon^2 R^2 ||\nabla W||^2,$$

which is small with ϵ.

$$B \leq \lambda(t)^{-2} (R\lambda(t))^2 ||v||^2_{L^{2^*}(|y| \geq \epsilon R \lambda(t))} \xrightarrow[t \to T_+(u_0)]{} 0,$$

(since $\lambda(t) \uparrow +\infty$ as $t \to T_+(u_0)$) using the compactness of $\overline{K}$. But then, $y_R(0) \leq C T_+(u_0) ||\nabla W||^2$, by the fundamental theorem of calculus. Thus, leting $R \to \infty$, we see that $u_0 \in L^2$, but then, using the conservation of the L^2 norm, we see that $||u_0||_{L^2} = ||u(T_+(u_0))||_{L^2} = 0$, so that $u_0 \equiv 0$.

Case 2: $T_+(u_0) = +\infty$ First note that the compactness of $\overline{K}$, together with $\lambda(t) \geq A_0 > 0$, gives that, given $\epsilon > 0$, there exists $R(\epsilon) > 0$ such that, for all $t \in [0, +\infty)$,

$$\int_{|x|>R(\epsilon)} |\nabla u|^2 + |u|^{2^*} + \frac{|u|^2}{|x|^2} \leq \epsilon.$$

Pick $\delta_0 > 0$ so that $E(u_0) \leq (1 - \delta_0)E(W)$. Recall that, by our variational estimates, we have that $\int |\nabla u(t)|^2 - |u(t)|^{2^*} \geq C_{\delta_0} ||\nabla u_0||^2_{L^2}$. If $||\nabla u_0||_{L^2} \neq 0$, using the smallness of tails, we see that , for $R > R_0$,

$$\int_{|x|<R} |\nabla u(t)|^2 - |u(t)|^{2^*} \geq C_{\delta_0} ||\nabla u_0||^2_{L^2}.$$

Choose now $\psi \in C_0^\infty$, radial, with $\psi(x) = |x|^2$ for $|x| \leq 1$, $\operatorname{supp} \psi \subset \{|x| \leq 2\}$. Define now

$$z_R(t) = \int |u(x,t)|^2 R^2 \psi(x/R)\, dx.$$

Similar computations to Glassey's blow-up proof give:

$$z'_R(t) = 2R \operatorname{Im} \int \overline{u} \nabla u \nabla \psi(x/R)$$

and

$$\begin{aligned} z''_R(t) &= 4 \sum_{l,j} \operatorname{Re} \int \partial_{x_l} \partial_{x_j} \psi(x/R) \partial_{x_l} u \partial_{x_j} \overline{u} \\ &- \frac{1}{R^2} \int \triangle^2 \psi(x/R) |u|^2 - \frac{4}{N} \int \triangle \psi(x/R) |u|^{2^*}. \end{aligned}$$

Note that $|z'_R(t)| \leq C_{\delta_0} R^2 ||\nabla u_0||^2$, by Cauchy–Schwartz, Hardy's inequality and our variational estimates. On the other hand,

$$\begin{aligned} z''_R(t) &\geq \left[\int_{|x|\leq R} |\nabla u(t)|^2 - |u(t)|^{2^*} \right] \\ &- C \left(\int_{R\leq |x| \leq 2R} |\nabla u(t)|^2 + \frac{|u|^2}{|x|^2} + |u(t)|^{2^*} \right) \\ &\geq C ||\nabla u_0||^2, \end{aligned}$$

for R large. Integrating in t, we obtain: $z'_R(t) - z'_R(0) \geq Ct ||\nabla u_0||^2$, but $|z'_R(t) - z'_R(0)| \leq 2CR^2 ||\nabla u_0||^2$, which is a contradiction for t large, proving Theorem 1, i).

Remark 9. In the defocusing case, the proof is easier since the variational estimates are not needed.

Remark 10. It is quite likely that for $N = 3$, examples similar to those by P. Raphäel [38] can be constructed, of radial data u_0, for which $T_+(u_0) < \infty$ and u blows-up exactly on a sphere.

We now turn to Theorem 2. We thus consider

$$\begin{cases} \partial_t^2 u - \triangle u = |u|^{4/N-2} u & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u\big|_{t=0} = u_0 \in \dot{\mathrm{H}}^1(\mathbb{R}^n) & \\ \partial_t u\big|_{t=0} = u_1 \in L^2(\mathbb{R}^n) & N \geq 3 \end{cases} \tag{7}$$

Recall that $W(x) = \left(1 + |x|^2/N(N-2)\right)^{-(N-2)/2}$ is a static solution that does not scatter. The general scheme of the proof is similar to the one for Theorem 1. We start out with a brief review of the "local Cauchy problem". We first consider the asociated linear problem,

$$\begin{cases} \partial_t^2 w - \triangle w = h \\ w\big|_{t=0} = w_0 \in \dot{\mathrm{H}}^1(\mathbb{R}^n) \\ \partial_t w\big|_{t=0} = w_1 \in L^2(\mathbb{R}^N) \end{cases} \tag{8}$$

As is well known (see [42] for instance), the solution is given by

$$\begin{aligned} w(x,t) &= \cos\left(t\sqrt{-\triangle}\right) w_0 + (-\triangle)^{-1/2} \sin\left(t\sqrt{-\triangle}\right) w_1 \\ &+ \int_0^t (-\triangle)^{-1/2} \sin\left((t-s)\sqrt{-\triangle}\right) h(s)\, ds \\ &= S(t)((w_0, w_1)) + \int_0^t (-\triangle)^{-1/2} \sin\left((t-s)\sqrt{-\triangle}\right) h(s)\, ds. \end{aligned}$$

The following are the relevant Strichartz estimates: for an interval $I \subset \mathbb{R}$, let

$$\|f\|_{S(I)} = \|f\|_{L_I^{2(N+1)/N-2} L_x^{2(N+1)/N-2}},$$

$$\|f\|_{W(I)} = \|f\|_{L_I^{2(N+1)/N-1} L_x^{2(N+1)/N-1}}.$$

Then (see [14], [23])

$$\begin{cases} \sup_t \|(w(t), \partial_t w(t))\|_{\dot{\mathrm{H}}^1 \times L^2} + \left\|D^{1/2} w\right\|_{W(-\infty,+\infty)} + \\ + \left\|\partial_t D^{-1/2} w\right\|_{W(-\infty,+\infty)} + \|w\|_{S(-\infty,+\infty)} + \\ + \|w\|_{L_t^{(N+2)/N-2} L_x^{2(N+2)/N-2}} \leq \\ \leq C\left\{\|(w_0, w_1)\|_{\dot{\mathrm{H}}^1 \times L^2} + \|w\|_{L_t^{2(N+1)/N+3} L_x^{2(N+1)/N+3}}\right\} \end{cases} \tag{9}$$

Because of the appearence of $D^{1/2}$ in these estimates, we also need to use the following version of the chain rule for fractional derivatives (see [26]).

Lemma 2. *Assume* $F \in C^2$*,* $F(0) = F'(0) = 0$ *and that for all* a*,* b *we have* $|F'(a+b)| \leq C\{|F'(a)| + |F'(b)|\}$ *and* $|F''(a+b)| \leq C\{|F''(a)| + |F''(b)|\}$*. Then, for* $0 < \alpha < 1$*,* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$*,* $\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$*, we have*

i) $\|D^\alpha F(u)\|_{L^p} \leq C\,\|F'(u)\|_{L^{p_1}}\,\|D^\alpha u\|_{L^{p_2}}$*,*

ii) $\|D^\alpha(F(u) - F(v))\|_{L^p} \leq C\left[\|F'(u)\|_{L^{p_1}} + \|F'(v)\|_{L^{p_1}}\right]\|D^\alpha(u-v)\|_{L^{p_2}}$

$$+C\left[\|F''(u)\|_{L^{r_1}} + \|F''(v)\|_{L^{r_1}}\right]\left[\|D^\alpha u\|_{L^{r_2}} + \|D^\alpha v\|_{L^{r_2}}\right]\|u-v\|_{L^{r_3}}.$$

Using (9) and this Lemma, one can now use the same argument as for (4) to obtain:

Theorem 7 ([14], [20], [41] and [23])**.** *Asume that*

$$(u_0, u_1) \in \dot{\mathrm{H}}^1 \times L^2, \quad \|(u_0, u_1)\|_{\dot{\mathrm{H}}^1 \times L^2} \leq A.$$

Then (for $3 \leq N \leq 6$*) there exists* $\delta = \delta(A) > 0$ *such that if* $\|S(t)(u_0, u_1)\|_{S(I)} \leq \delta$*,* $0 \in \mathring{I}$*, there exists a unique solution to (7) in* $\mathbb{R}^N \times I$*, with* $(u, \partial_t u) \in C(I; \dot{\mathrm{H}}^1 \times L^2)$ *and* $\left\|D^{1/2}u\right\|_{W(I)} + \left\|\partial_t D^{-1/2}u\right\|_{W(I)} < \infty$*,* $\|u\|_{S(I)} \leq 2\delta$*. Moreover, the mapping* $(u_0, u_1) \in \dot{\mathrm{H}}^1 \times L^2 \to (u, \partial_t u) \in C(I; \dot{\mathrm{H}}^1 \times L^2)$ *is Lipschitz.*

Remark 11. Again, using (9), if $\|(u_0, u_1)\|_{\dot{\mathrm{H}}^1 \times L^2} \leq \tilde{\delta}$, the hypothesis of the Theorem is verified for $I = (-\infty, +\infty)$. Moreover, given $(u_0, u_1) \in \dot{\mathrm{H}}^1 \times L^2$, we can find $\mathring{I} \ni 0$ so that the hypothesis is verified on I. One can then define a maximal interval of existence $I = (-T_-(u_0, u_1), T_+(u_0, u_1))$, similarly to the case of (4). We also have the "standard finite time blow-up criterion": if $T_+(u_0, u_1) < \infty$, then $\|u\|_{S(0, T_+(u_0, u_1))} = +\infty$. Also, if $T_+(u_0, u_1) = +\infty$, u scatters at $+\infty$ (i.e. $\exists (u_0^+, u_1^+) \in \dot{\mathrm{H}}^1 \times L^2$ such that $\left\|(u(t), \partial_t u(t)) - S(t)(u_0^+, u_1^+)\right\|_{\dot{\mathrm{H}}^1 \times L^2} \xrightarrow[t \uparrow +\infty]{} 0$) if and only if $\|u\|_{S(0, +\infty)} < +\infty$. Moreover, for $t \in I$, we have

$$E((u_0, u_1)) = \frac{1}{2}\int |\nabla u_0|^2 + \frac{1}{2}\int u_1^2 - \frac{1}{2^*}\int |u_0|^{2^*} = E((u(t), \partial_t u(t))).$$

It turns out that for (7) there is another very important conserved quantity in the energy space, namely momentum. This is crucial for us to be able to treat non-radial data. This says that, for $t \in I$, $\int \nabla u(t) \cdot \partial_t u(t) = \int \nabla u_0 \cdot u_1$. Finally, the analog of the "Perturbation Theorem" also holds in this context (see [22]). All the corollaries of the Perturbation Theorem also hold.

Remark 12 (Finite speed of propagation). Recall that if $R(t)$ is the forward fundamental solution for the linear wave equation, the solution for (8) is given by (see [42])

$$w(t) = \partial_t R(t) * w_0 + R(t) * w_1 - \int_0^t R(t-s) * h(s)\,ds,$$

where $*$ stands for convolution in the x variable. The finite speed of propagation is the statement that $\operatorname{supp} R(\cdot,t)$, $\operatorname{supp} \partial_t R(\cdot,t) \subset \overline{B(0,t)}$. Thus, if $\operatorname{supp} w_0 \subset {}^C B(x_0,a)$, $\operatorname{supp} w_1 \subset {}^C B(x_0,a)$, $\operatorname{supp} h \subset {}^C[\bigcup_{0\le t\le a} B(x_0,a-t)\times\{t\}]$, then $w \equiv 0$ on $\bigcup_{0\le t\le a} B(x_0,a-t)\times\{t\}$. This has important consequences for solutions of (7). If $(u_0,u_1) \equiv (u_0',u_1')$ on $B(x_0,a)$, then the corresponding solutions agree on $\bigcup_{0\le t\le a} B(x_0,a-t)\times\{t\} \cap \mathbb{R}^N \times (I\cap I')$.

We now proceed with the proof of Theorem 2. As in the case of (4), the proof is broken up in three steps.

Step1: Variational estimates Here these are immediate from the corresponding ones in (4). The summary is: (we use the notation $\mathcal{E}(v) = \frac{1}{2}\int|\nabla v|^2 - \frac{1}{2^*}\int|v|^{2^*}$).

Lemma 3. *Let $(u_0,u_1) \in \dot{\mathrm{H}}^1 \times L^2$ be such that $E((u_0,u_1)) \le (1-\delta_0)E((W,0))$, $||\nabla u_0|^2 < ||\nabla W||^2$. Let u be the corresponding solution of (7), with maximal interval I. Then, there exists $\overline{\delta} = \overline{\delta}(\delta_0) > 0$ such that, for $t \in I$ we have*

i) $||\nabla u(t)|| \le (1-\overline{\delta})||\nabla W||$

ii) $\displaystyle\int |\nabla u(t)|^2 - |u(t)|^{2^*} \ge \overline{\delta}\int|\nabla u(t)|^2$

iii) $\mathcal{E}(u(t)) \ge 0$ *(and here $E((u,\partial_t u)) \ge 0$)*

iv) $E((u,\partial_t u)) \approx \|(u(t),\partial_t u(t))\|^2_{\dot{\mathrm{H}}^1\times L^2} \approx \|(u_0,u_1)\|^2_{\dot{\mathrm{H}}^1\times L^2}$, *with comparability constants depending only on δ_0.*

Remark 13. If $E((u_0,u_1)) \le (1-\delta_0)E((W,0))$, $||\nabla u_0||^2 > ||\nabla W||^2$, then for $t\in I$, $\|\nabla u(t)\|^2 \ge (1+\overline{\delta})\|\nabla W\|^2$. This follows from the corresponding result for (4).

We now turn to the proof of ii) in Theorem 2. We will do it for the case when $\|u_0\|_{L^2} < \infty$. For the general case, see [23]. We know that, in the situation of ii), we have

$$\int|\nabla u(t)|^2 \ge (1+\overline{\delta})\int|\nabla W|^2, \quad t\in I$$

$$E((W,0)) \ge E((u(t),\partial_t u)) + \tilde{\delta}_0.$$

Thus,

$$\frac{1}{2^*}\int|u(t)|^{2^*} \ge \frac{1}{2}\int(\partial_t u(t))^2 + \frac{1}{2}\int|\nabla u(t)|^2 - E((W,0)) + \tilde{\delta}_0,$$

so that

$$\int|u(t)|^{2^*} \ge \frac{N}{N-2}\int(\partial_t u(t))^2 + \frac{N}{N-2}\int|\nabla u(t)|^2 - 2^*E((W,0)) + 2^*\tilde{\delta}_0.$$

Let $y(t) = \int|u(t)|^2$, so that $y'(t) = 2\int u(t)\partial_t u(t)$. A simple calculation gives

$$y''(t) = 2\int\left\{(\partial_t u)^2 - |\nabla u(t)|^2 + |u(t)|^{2^*}\right\}.$$

Thus,

$$\begin{aligned} y''(t) &\geq 2\int(\partial_t u)^2 + \frac{2N}{N-2}\int(\partial_t u)^2 - 2\cdot 2^* E((W,0)) + \\ &+ \tilde{\tilde{\delta}}_0 + \frac{2N}{N-2}\int|\nabla u(t)|^2 - 2\int|\nabla u(t)|^2 = \\ &= \frac{4(N-1)}{N-2}\int(\partial_t u)^2 + \frac{4}{N-2}\int|\nabla u(t)|^2 - \\ &- \frac{4}{N-2}\int|\nabla W|^2 + \tilde{\tilde{\delta}}_0 \geq \\ &\geq \frac{4(N-1)}{N-2}\int(\partial_t u)^2 + \tilde{\tilde{\delta}}_0. \end{aligned}$$

If $I \cap [0,+\infty) = [0,+\infty)$, there exists $t_0 > 0$ so that $y'(t_0) > 0$, $y'(t) > 0$, $t > t_0$. For $t > t_0$ we have

$$y(t)y''(t) \geq \frac{4(N-1)}{N-2}\int(\partial_t u)^2\int u^2 \geq \left(\frac{N-1}{N-2}\right)y'(t)^2,$$

so that

$$\frac{y''(t)}{y'(t)} \geq \left(\frac{N-1}{N-2}\right)\frac{y'(t)}{y(t)},$$

or

$$y'(t) \geq C_0 y(t)^{(N-1)/(N-2)}, \text{ for } t > t_0.$$

But, since $N-1/N-2 > 1$, this leads to finite time blow-up, a contradiction.

We next turn to the proof of i) in Theorem 2.

Step 2: Concentration-compactness procedure. Here we proceed initially in an identical manner as in the case of (4), replacing the "profile decomposition" of Keraani [27], with the corresponding one for the wave equation, due to Bahouri–Gérard [2]. Thus, arguing by contradiction, we find a number E_c, with $0 < \eta_0 \leq E_c < E((W,0))$ with the property that if $E((u_0,u_1)) < E_c$, $\|\nabla u_0\|^2 < \|\nabla W\|^2$, $\|u\|_{S(I)} < \infty$ and E_c is optimal with this property. We will see that this leads to a contradiction. As for (4), we have:

Proposition 3. *There exists*

$$(u_{0,c}, u_{1,c}) \in \dot{\mathrm{H}}^1 \times L^2, \quad \|\nabla u_{0,c}\|^2 < \|\nabla W\|^2, \quad E((u_{0,c}, u_{1,c})) = E_c$$

and such that for the corresponding solution u_c on (7) we have $\|u_c\|_{S(I)} = +\infty$.

Proposition 4. *For any u_c as in Proposition 3, with (say) $\|u_c\|_{S(I_+)} = +\infty$, $I_+ = I \cap [0, +\infty)$, there exists $x(t) \in \mathbb{R}^N$, $\lambda(t) \in \mathbb{R}^+$, $t \in I_+$, such that*

$$K = \left\{ v(x,t) \quad = \quad \left(\frac{1}{\lambda(t)^{N-2/2}} u_c \left(\frac{x - x(t)}{\lambda(t)}, t \right), \right. \right.$$
$$\left. \left. \frac{1}{\lambda(t)^{N/2}} \partial_t u_c \left(\frac{x - x(t)}{\lambda(t)}, t \right) \right) : t \in I_+ \right\}$$

has compact closure in $\dot{\mathrm{H}}^1 \times L^2$.

Remark 14. As in the case of (4), in Proposition 4 we can construct $\lambda(t)$, $x(t)$ continuous in $[0, T_+((u_{0,c}, u_{1,c})))$. Moreover, by scaling and compactness of $\overline{K}$, if $T_+((u_{0,c}, u_{1,c})) < \infty$, we have $\lambda(t) \geq C_0(K)/(T_+((u_{0,c}, u_{1,c})) - t)$. Also, if $T_+((u_{0,c}, u_{1,c})) = +\infty$, we can always find another (possibly different)critical element v_c, with a corresponding $\tilde{\lambda}$ so that $\tilde{\lambda}(t) \geq A > 0$, for $t \in [0, T_+((v_{0,c}, v_{1,c})))$, using the compactness of $\overline{K}$. We can also find g decreasing, $g : (0, E_c] \to [0, +\infty)$ so that if $\|\nabla u_0\|^2 < \|\nabla W\|^2$ and $E((u_{0,c}, u_{1,c})) \leq E_c - \eta$, then $\|u\|_{S(-\infty, +\infty)} \leq g(\eta)$.

Up to here, we have used, in Step 2, only Step 1 and "general arguments". To proceed further we need to use specific features of (7) to establish further properties of critical elements.

The first one is a consequence of the finite speed of propagation and the compactness of $\overline{K}$.

Lemma 4. *Let u_c be a critical element as in Proposition 4, with $T_+((u_{0,c}, u_{1,c})) < +\infty$. (We can assume, by scaling, that $T_+((u_{0,c}, u_{1,c})) = 1$). Then, there exists $\overline{x} \in \mathbb{R}^N$ such that* $\operatorname{supp} u_c(\cdot, t), \operatorname{supp} \partial_t u_c(\cdot, t) \subset B(\overline{x}, 1 - t)$, $0 < t < 1$.

In order to prove this Lemma, we will need the following consequence of the finite speed of propagation:

Remark 15. Let $(u_0, u_1) \in \dot{\mathrm{H}}^1 \times L^2$, $\|(u_0, u_1)\|_{\dot{\mathrm{H}}^1 \times L^2} \leq A$. If for some $M > 0$, $\epsilon > 0$, $0 < \epsilon < \epsilon_0(A)$ we have

$$\int_{|x| \geq M} |\nabla u_0|^2 + |u_1|^2 + \frac{|u_0|^2}{|x|^2} \leq \epsilon,$$

then for $0 < t < T_+(u_0, u_1)$ we have

$$\int_{|x| \geq \frac{3}{2}M + t} |\nabla u(t)|^2 + |\partial_t u(t)|^2 + |u(t)|^{2^*} + \frac{|u(t)|^2}{|x|^2} \leq C\epsilon.$$

Indeed, choose $\psi_M \in C^\infty$, $\psi_M \equiv 1$ for $|x| \geq \frac{3}{2}M$, with $\psi_M \equiv 0$ for $|x| \leq M$. Let $u_{0,M} = u_0 \psi_M$, $u_{1,M} = u_1 \psi_M$. From our assumptions, we have $\|(u_{0,M}, u_{1,M})\|_{\dot{\mathrm{H}}^1 \times L^2} \leq C\epsilon$. If $C\epsilon_0 < \tilde{\delta}$, where $\tilde{\delta}$ is as in the "local Cauchy theory", the corresponding solution of (7), u_M has maximal interval $(-\infty, +\infty)$ and

$\sup_{t\in(-\infty,+\infty)} \|(u_M(t), \partial_t u_M(t))\|_{\dot{H}^1\times L^2} \le 2C\epsilon$. But, by finite speed of propagation, $u_M \equiv u$ for $|x| \ge \frac{3}{2}M + t$, $t \in [0, T_+(u_0, u_1))$, which shows the Remark.

We turn to the proof of the Lemma. Recall that $\lambda(t) \ge C_0(K)/(1-t)$. We claim that, for any $R_0 > 0$,

$$\lim_{t\uparrow 1} \int_{|x+x(t)/\lambda(t)|\ge R_0} |\nabla u_c(x,t)|^2 + |\partial_t u_c(x,t)|^2 + \frac{|u_c(x,t)|^2}{|x|^2} = 0.$$

Indeed, if $\vec{v}(x,t) = \frac{1}{\lambda(t)^{N/2}} \left(\nabla u_c \left(\frac{x-x(t)}{\lambda(t)}, t \right), \partial_t u_c \left(\frac{x-x(t)}{\lambda(t)}, t \right) \right)$,

$$\int_{|x+x(t)/\lambda(t)|\ge R_0} |\nabla u_c(x,t)|^2 + |\partial_t u_c(x,t)|^2 = \int_{|y|\ge \lambda(t) R_0} |\vec{v}(x,t)|^2 dy \underset{t\uparrow 1}{\longrightarrow} 0,$$

because of the compactness of $\overline{K}$ and the fact that $\lambda(t) \to +\infty$ as $t \to 1$. Because of this fact, using the Remark, backward in time, we have, for each $s \in [0,1)$, $R_0 > 0$

$$\lim_{t\uparrow 1} \int_{|x+x(t)/\lambda(t)|\ge \frac{3}{2}R_0 + (t-s)} |\nabla u_c(x,s)|^2 + |\partial_t u_c(x,s)|^2 = 0.$$

We next show that $|x(t)/\lambda(t)| \le M$, $0 \le t < 1$. If not, we can find $t_n \uparrow 1$ so that$|x(t_n)/\lambda(t_n)| \to +\infty$. Then, for $R > 0$, $\{|x| \le R\} \subset \{|x + x(t_n)/\lambda(t_n)| \ge \frac{3}{2}R + t_n\}$, for n large, so that, passing to the limit in n, for $s = 0$, we obtain

$$\int_{|x|\le R} |\nabla u_{0,c}|^2 + |u_{1,c}|^2 = 0,$$

a contradiction.

Finally, pick $t_n \uparrow 1$ so that $x(t_n)/\lambda(t_n) \to -\overline{x}$. Observe that, for every $\eta_0 > 0$, for n large enough, for all $s \in [0,1)$, $\{|x - \overline{x}| \ge 1 + \eta_0 - s\} \subset \{|x + x(t_n)/\lambda(t_n)| \ge \frac{3}{2}R_0 + (t_n - s)\}$, for some $R_0 = R_0(\eta_0) > 0$. From this we conclude that

$$\int_{|x-x_0|\ge 1+\eta_0-s} |\nabla u(x,s)|^2 + |\partial_s u(x,s)|^2 dx = 0,$$

which gives the claim.

Note that, after translation we can asume that $\overline{x} = 0$. We next turn to a result which is fundamental for us to be able to treat non-radial data.

Theorem 8. *Let $(u_{0,c}, u_{1,c})$ be as in Proposition 4, with $\lambda(t)$, $x(t)$ continuous. Assume that either $T_+(u_{0,c}, u_{1,c}) < \infty$ or $T_+(u_{0,c}, u_{1,c}) = +\infty$, $\lambda(t) \ge A_0 > 0$. Then*

$$\int \nabla u_{0,c} \cdot u_{1,c} = 0.$$

In order to carry out the proof of this Theorem, a further linear estimate is needed:

Lemma 5. *Let w solve the linear wave equation*

$$\begin{cases} \partial_t^2 w - \triangle w = h \in L^1_t L^2_x(\mathbb{R}^{N+1}) \\ w\big|_{t=0} = w_0 \in \dot{\mathrm{H}}^1(\mathbb{R}^n) \\ \partial_t w\big|_{t=0} = w_1 \in L^2(\mathbb{R}^N) \end{cases}$$

Then, for $|a| \leq 1/4$, we have

$$\sup_t \left\| \left(\nabla w \left(\frac{x_1 - at}{\sqrt{1-a^2}}, x', \frac{t - ax_1}{\sqrt{1-a^2}} \right), \partial_t w \left(\frac{x_1 - at}{\sqrt{1-a^2}}, x', \frac{t - ax_1}{\sqrt{1-a^2}} \right) \right) \right\|_{L^2(dx_1 dx')}$$
$$\leq C \left\{ \|w_0\|_{\dot{\mathrm{H}}^1} + \|w_1\|_{L^2} + \|h\|_{L^1_t L^2_x} \right\}.$$

The simple proof is omitted. See [23] for the details. Note that if u is a solution of (7), with maximal interval I and $I' \subset\subset I$, $u \in L_{I'}^{(N+2)/N-2} L_x^{2(N+2)/N-2}$, and since $\frac{4}{N-2} + 1 = \frac{N+2}{N-2}$, $|u|^{4/N-2} u \in L^1_{I'} L^2_x$. Thus, the conclusion of the Lemma applies, provided the integration is restricted to $\left(\frac{x_1 - at}{\sqrt{1-a^2}}, x', \frac{t-ax_1}{\sqrt{1-a^2}} \right) \in \mathbb{R}^N \times I'$.

Sketch of the proof of the Theorem. Assume first that $T_+(u_{0,c}, u_{1,c}) = 1$. Assume, to argue by contradiction, that (say) $\int \partial_{x_1}(u_{0,c}) u_{1,c} = \gamma > 0$. Recall that, in this situation, $\operatorname{supp} u_c, \partial_t u_c \subset B(0, 1-t)$, $0 < t < 1$. For convenience, set $u(x,t) = u_c(x, 1+t)$, $-1 < t < 0$, which is supported in $B(0, |t|)$. For $0 < a < 1/4$, we consider the Lorentz transformation

$$z_a(x_1, x', t) = u \left(\frac{x_1 - at}{\sqrt{1-a^2}}, x', \frac{t - ax_1}{\sqrt{1-a^2}} \right),$$

and we fix our attention on $-1/2 \leq t < 0$. In that region, the previous Lemma and the comment following show, in conjunction with the support property of u, that z_a is a solution in the energy space of (7). An easy calculation shows that $\operatorname{supp} z_a(\cdot, t) \subset B(0, |t|)$, so that 0 is the final time of existence for z_a. A lengthy calculation shows that

$$\lim_{a \downarrow 0} \frac{E((z_a(\cdot, -1/2), \partial_t z_a(\cdot, -1/2))) - E((u_{0,c}, u_{1,c}))}{a} = -\gamma$$

and that, for some $t_0 \in [-1/2, -1/4]$, $\int |\nabla z_a(t_0)|^2 < \int |\nabla W|^2$, for a small (by integration in t_0 and a change of variables, together with the variational estimates for u_c). But, since $E((u_{0,c}, u_{1,c})) = E_c$, for a small this contradicts the definition of E_c, since the final time of existence of z_a is finite.

In the case when $T_+(u_{0,c}, u_{1,c}) = +\infty$, $\lambda(t) \geq A_0 > 0$, the finiteness of the energy of z_a is unclear, because of the lack of the support property. We instead do a renormalization. We first rescale u_c and consider, for R large, $u_R(x,t) = R^{N-2/2}u_c(Rx,Rt)$, and for a small

$$z_{a,R}(x_1,x',t) = u_R\left(\frac{x_1-at}{\sqrt{1-a^2}}, x', \frac{t-ax_1}{\sqrt{1-a^2}}\right).$$

We assume, as before, that $\int \partial_{x_1}(u_{0,c})u_{1,c} = \gamma > 0$ and hope to obtain a contradiction. We prove, by integration in $t_0 \in (1,2)$ that if $h(t_0) = \theta(x)z_{a,R}(x_1,x',t_0)$, with θ a fixed cut-off function, for some a_1 small and R large, we have, for some $t_0 \in (1,2)$ that

$$E((h(t_0), \partial_t h(t_0))) < E_c - \frac{1}{2}\gamma a_1$$

and

$$\int |\nabla h(t_0)|^2 < \int |\nabla W|^2.$$

We then let v be the solution of (7), with data $h(\cdot,t_0)$. By the properties of E_c, we know that $\|v\|_{S(-\infty,+\infty)} \le g(\frac{1}{2}\gamma a_1)$, for R large. But, since $\|u_c\|_{S(0,+\infty)} = +\infty$, we have that

$$\|u_R\|_{L^{2(N+1)/N-2}_{[0,1]} L^{2(N+1)/N-2}_{\{|x|<1\}}} \xrightarrow[R\to\infty]{} \infty.$$

But, by finite speed of propagation, we have that $v = z_{a,R}$ on a large set and after a change of variables to undo the Lorentz transformation, we reach a contradiction from these two facts. □

From all this we see that, to prove Theorem 2, it suffices to show:

Step 3: Rigidity Theorem

Theorem 9 (Rigidity). *Suppose that $E((u_0,u_1)) < E((W,0))$, $\int |\nabla u_0|^2 < \int |\nabla W|^2$, u is the corresponding solution of (7), and we let $I_+ = [0, T_+((u_0,u_1)))$. Assume that:*

a) $\int \nabla u_0 u_1 = 0$

b) There exist $x(t)$, $\lambda(t)$, $t \in [0, T_+((u_0,u_1)))$ such that

$$K = \left\{ v(x,t) = \left(\frac{1}{\lambda(t)^{N-2/2}}u_c\left(\frac{x-x(t)}{\lambda(t)},t\right), \frac{1}{\lambda(t)^{N/2}}\partial_t u_c\left(\frac{x-x(t)}{\lambda(t)},t\right)\right) : t \in I_+ \right\}$$

has compact closure in $\dot{\mathrm{H}}^1 \times L^2$.

c) $x(t)$, $\lambda(t)$ are continuous, $\lambda(t) > 0$. If $T_+(u_0,u_1) < \infty$, we have $\lambda(t) \ge C/(T_+ - t)$, $\operatorname{supp} u, \partial_t u \subset B(0, T_+ - t)$, if $T_+(u_0,u_1) = +\infty$, we have $x(0) = 0$, $\lambda(0) = 1$, $\lambda(t) \ge A_0 > 0$.

Then, $T_+(u_0,u_1)=+\infty$, $u\equiv 0$.

Clearly this Rigidity Theorem provides the contradiction that concludes the proof of Theorem 2.

Proof of the Rigidity Theorem. For the proof we need some known identities (see [42], [23]).

Lemma 6. *Let*

$$r(R)=r(t,R)=\int_{|x|\geq R}\left\{|\nabla u|^2+|\partial_t u|^2+|u|^{2^*}+\frac{|u|^2}{|x|^2}\right\}\,dx.$$

Let u *be a solution of (7),* $t\in I$, $\phi_R(x)=\phi(x/R)$, $\psi_R(x)=x\phi(x/R)$, *where* ϕ *is in* $C_0^\infty(B_2)$, $\phi\equiv 1$ *on* $|x|\leq 1$. *Then*

i) $$\partial_t\left(\int\psi_R\nabla u\partial_t u\right)=-\frac{N}{2}\int(\partial_t u)^2+\frac{N-2}{2}\int\left[|\nabla u|^2-|u|^{2^*}\right]+\mathcal{O}(r(R))$$

ii) $$\partial_t\left(\int\phi_R\nabla u\partial_t u\right)=\int(\partial_t u)^2-\int|\nabla u|^2+\int|u|^{2^*}+\mathcal{O}(r(R))$$

iii) $$\partial_t\left(\int\psi_R\left\{\frac{1}{2}|\nabla u|^2+\frac{1}{2}(\partial_t u)^2-\frac{1}{2^*}|u|^{2^*}\right\}\right)=-\int\nabla u\partial_t u+\mathcal{O}(r(R)).$$

We start out the proof of case 1, $T_+((u_0,u_1))=+\infty$, by observing that, if $(u_0,u_1)\neq(0,0)$ and $E=E((u_0,u_1))$, then, from our variational estimates, $E>0$ and

$$\sup_{t>0}\|(\nabla u(t),\partial_t u(t))\|_{\dot{\mathrm{H}}^1\times L^2}\leq CE.$$

We also have

$$\int|\nabla u(t)|^2-|u(t)|^{2^*}\geq C\int|\nabla u(t)|^2,\quad t>0$$

and

$$\frac{1}{2}\int(\partial_t u(t))^2+\frac{1}{2}\int\left[|\nabla u(t)|^2-|u(t)|^{2^*}\right]\geq CE,\quad t>0.$$

The compactness of $\overline{K}$ and the fact that $\lambda(t)\geq A_0>0$ show that, given $\epsilon>0$, we can find $R_0(\epsilon)>0$ so that, for all $t>0$ we have

$$\int_{\left|x+\frac{x(t)}{\lambda(t)}\right|\geq R(\epsilon)}|\partial_t u|^2+|\nabla u|^2+\frac{|u|^2}{|x|^2}+|u|^{2^*}\leq\epsilon E.$$

The proof of this case is accomplished through two lemmas.

Lemma 7. *There exists* $\epsilon_1>0$, $C>0$ *such that, if* $0<\epsilon<\epsilon_1$, *if* $R>2R_0(\epsilon)$, *there exists* $t_0=t_0(R,\epsilon)$ *with* $0<t_0\leq CR$, *such that for* $0<t<t_0$, *we have* $\left|\frac{x(t)}{\lambda(t)}\right|<R-R_0(\epsilon)$ *and* $\left|\frac{x(t)}{\lambda(t)}\right|=R-R_0(\epsilon)$.

Note that in the radial case, since we can take $x(t) \equiv 0$, a contradiction follows directly from Lemma 7. This will be the analog of the local virial identity proof for the corresponding case of (4). For the non-radial case we also need:

Lemma 8. *There exists $\epsilon_2 > 0$, $R_1(\epsilon) > 0$, $C_0 > 0$, so that if $R > R_1(\epsilon)$, for $0 < \epsilon < \epsilon_2$, we have $t_0(R, \epsilon) \geq C_0 R/\epsilon$, where t_0 is as in Lemma 7.*

From Lemma 7 and Lemma 8 we have, for $0 < \epsilon < \epsilon_1$, $R > 2R_0(\epsilon)$, $t_0(R, \epsilon) \leq CR$, while for $0 < \epsilon < \epsilon_2$, $R > R_1(\epsilon)$, $t_0(R, \epsilon) \geq C_0 R/\epsilon$. This clearly is a contradiction for ϵ small.

Proof of Lemma 7. Since $x(0) = 0$, $\lambda(0) = 1$, if not, we have for all $0 < t < CR$, with C large, that $\left|\frac{x(t)}{\lambda(t)}\right| < R - R_0(\epsilon)$. Let

$$z_R(t) = \int \psi_R \nabla u \partial_t u + \left(\frac{N}{2} - \frac{1}{2}\right) \int \phi_R u \partial_t u.$$

Then,

$$z_R'(t) = -\frac{1}{2} \int (\partial_t u)^2 - \frac{1}{2} \int \left[|\nabla u|^2 - |u|^{2^*}\right] + \mathcal{O}(r(R)).$$

But, for $|x| > R$, $0 < t < CR$, we have $\left|x + \frac{x(t)}{\lambda(t)}\right| \geq R_0(\epsilon)$ so that $|r(R)| \leq \tilde{C}\epsilon E$. Thus, for ϵ small, $z_R'(t) \leq -\tilde{\tilde{C}}E/2$. By our variational estimates, we also have $|z_R(T)| \leq C_1 RE$. Integrating in t we obtain $CR\tilde{\tilde{C}}E/2 \leq 2C_1 RE$, which is a contradiction for C large. □

Proof of Lemma 8. For $0 \leq t \leq t_0$, set

$$y_R(t) = \int \psi_R \left\{\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{2^*}|u|^{2^*}\right\}.$$

For $|x| > R$, $\left|x + \frac{x(t)}{\lambda(t)}\right| \geq R_0(\epsilon)$, so that, since $\int \nabla u_0 u_1 = 0 = \int \nabla u(t) \partial_t u(t)$, $y'(R) = \mathcal{O}(r(R))$, and hence $|y_R(t_0) - y_R(0)| \leq \tilde{C}\epsilon E t_0$. But,

$$|y_R(0)| \leq \tilde{C}R_0(\epsilon)E + \mathcal{O}(Rr(R_0(\epsilon))) \leq \tilde{C}E[R_0(\epsilon) + \epsilon R].$$

Also,

$$\begin{aligned} |y_R(t_0)| \quad &\geq \quad \left|\int_{\left|x + \frac{x(t_0)}{\lambda(t_0)}\right| \leq R_0(\epsilon)} \psi_R \left\{\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{2^*}|u|^{2^*}\right\}\right| - \\ &- \quad \left|\int_{\left|x + \frac{x(t_0)}{\lambda(t_0)}\right| > R_0(\epsilon)} \psi_R \left\{\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{2^*}|u|^{2^*}\right\}\right|. \end{aligned}$$

In the first integral, $|x| \leq R$, so that $\psi_R(x) = x$. The second integral is bounded by $MR\epsilon E$. Thus,

$$|y_R(t_0)| \geq \left| \int_{\left|x+\frac{x(t_0)}{\lambda(t_0)}\right| \leq R_0(\epsilon)} x \left\{ \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{2^*}|u|^{2^*} \right\} \right| - MR\epsilon E.$$

The integral on the right equals

$$\begin{aligned} - \quad & \frac{x(t_0)}{\lambda(t_0)} \int_{\left|x+\frac{x(t_0)}{\lambda(t_0)}\right| \leq R_0(\epsilon)} \left\{ \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{2^*}|u|^{2^*} \right\} + \\ + \quad & \int_{\left|x+\frac{x(t_0)}{\lambda(t_0)}\right| \leq R_0(\epsilon)} \left(x + \frac{x(t_0)}{\lambda(t_0)} \right) \left\{ \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{2^*}|u|^{2^*} \right\}, \end{aligned}$$

so that its absolute value is greater than or equal to

$$(R_0 - R_0(\epsilon))E - \tilde{C}(R - R_0(\epsilon))\epsilon E - \tilde{C}R_0(\epsilon)E,$$

thus,

$$|y_R(t_0)| \geq E(R - R_0(\epsilon))[1 - \tilde{C}\epsilon] - \tilde{C}R_0(\epsilon)E - MR\epsilon E \geq ER/4,$$

for R large, ϵ small. But then, $ER/4 - \tilde{C}E[R_0(\epsilon) + \epsilon R] \leq \tilde{C}\epsilon E t_0$, which yields the Lemma for ϵ small, R large. □

We next turn to the case 2, $T_+((u_0, u_1)) = 1$, with $\operatorname{supp} u, \partial_t u \subset B(0, 1-t)$, $\lambda(t) \geq C/1-t$. For (7) we canot use the conservation of the L^2 norm as in the (4) case and a new approach is needed. The first step is:

Lemma 9. *Let u be as in the rigidity theorem, with $T_+((u_0, u_1)) = 1$. Then, there exists $C > 0$ so that $\lambda(t) \leq C/1-t$.*

Proof. If not, we can find $t_n \uparrow 1$, so that $\lambda(t_n)(1-t_n) \to +\infty$. Let

$$z(t) = \int x \nabla u \partial_t u + \left(\frac{N}{2} - \frac{1}{2} \right) \int u \partial_t u,$$

where we recall that z is well defined since $\operatorname{supp} u, \partial_t u \subset B(0, 1-t)$. Then, for $0 < t < 1$, we have

$$z'(t) = -\frac{1}{2} \int (\partial_t u)^2 - \frac{1}{2} \int |\nabla u|^2 - |u|^{2^*}.$$

By our variational estimates, $E((u_0, u_1)) = E > 0$ and

$$\sup_{0<t<1} \|(u(t), \partial_t u)\|_{\dot{\mathrm{H}}^1 \times L^2} \leq CE$$

and $z'(t) \le -CE$, for $0 < t < 1$. From the support properties of u, it is easy to see that $\lim_{t\uparrow 1} z(t) = 0$, so that, integrating in t we obtain

$$z(t) \ge CE(1-t), \quad 0 \le t < 1.$$

We will next show that $z(t_n)/1 - t_n \xrightarrow[n\to\infty]{} 0$, yielding a contradiction. Because $\int \nabla u(t)\partial_t u(t) = 0$, $0 < t < 1$, we have

$$\frac{z(t_n)}{1-t_n} = \int \frac{(x + x(t_n)/\lambda(t_n))\nabla u \partial_t u}{1-t_n} + \left(\frac{N}{2} - \frac{1}{2}\right) \int \frac{u\partial_t u}{1-t_n}.$$

Note that, for $\epsilon > 0$ given, we have

$$\int_{\left|x+\frac{x(t_n)}{\lambda(t_n)}\right| \le \epsilon(1-t_n)} \left|x + \frac{x(t_n)}{\lambda(t_n)}\right| |\nabla u(t_n)||\partial_t u(t_n)| + |u(t_n)||\partial_t u(t_n)| \le C\epsilon E(1-t_n).$$

Next we will show that $|x(t_n)/\lambda(t_n)| \le 2(1-t_n)$. If not $B(-x(t_n)/\lambda(t_n), (1-t_n)) \cap B(0, (1-t_n)) = \emptyset$, so that

$$\int_{B(-x(t_n)/\lambda(t_n),(1-t_n))} |\nabla u(t_n)|^2 + |\partial_t u(t_n)|^2 = 0,$$

while

$$\int_{\left|x+\frac{x(t_n)}{\lambda(t_n)}\right| \ge (1-t_n)} |\nabla u(t_n)|^2 + |\partial_t u(t_n)|^2 = \int_{|y|\ge \lambda(t_n)(1-t_n)} \left|\nabla u\left(\frac{y-x(t_n)}{\lambda(t_n)}, t_n\right)\right|^2 + \\ + \left|\partial_t u\left(\frac{y-x(t_n)}{\lambda(t_n)}, t_n\right)\right|^2 \frac{dy}{\lambda(t_n)^N} \xrightarrow[n\to\infty]{} 0,$$

which contradicts $E > 0$. Then

$$\begin{aligned} \frac{1}{1-t_n} \int_{\left|x+\frac{x(t_n)}{\lambda(t_n)}\right| \ge \epsilon(1-t_n)} & \left|x + \frac{x(t_n)}{\lambda(t_n)}\right| |\nabla u(t_n)||\partial_t u(t_n)| \le \\ \le \ & 3 \int_{\left|x+\frac{x(t_n)}{\lambda(t_n)}\right| \ge \epsilon(1-t_n)} |\nabla u(t_n)||\partial_t u(t_n)| = \\ = \ & 3 \int_{|y|\ge \epsilon(1-t_n)\lambda(t_n)} \left|\nabla u\left(\frac{y-x(t_n)}{\lambda(t_n)}, t_n\right)\right| \left|\partial_t u\left(\frac{y-x(t_n)}{\lambda(t_n)}, t_n\right)\right| \frac{dy}{\lambda(t_n)^N} \\ & \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

because of the compactness of $\overline{K}$ and the fact that $\lambda(t_n)(1-t_n) \to \infty$, Arguing similarly for $\int \frac{u\partial_t u}{1-t_n}$, using Hardy's inequality (centered at $-x(t_n)/\lambda(t_n)$), the proof is concluded. □

Proposition 5. *Let u be as in the rigidity theorem, with $T_+((u_0,u_1)) = 1$, $\operatorname{supp} u$, $\partial_t u \subset B(0, 1-t)$. Then,*

$$K = \Big((1-t)^{N-2/2} u((1-t)x, t), (1-t)^{N-2/2} \partial_t u((1-t)x, t)\Big)$$

is precompact in $\dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$.

Proof.

$$\Big\{\vec{v}(x,t) = (1-t)^{\frac{N}{2}} \left(\nabla u((1-t)(x-x(t)),t), \partial_t u((1-t)(x-x(t)),t)\right), 0 \le t < 1\Big\}$$

has compact closure in $L^2(\mathbb{R}^N)^{N+1}$, since we have $c_0 \le (1-t)\lambda(t) \le c_1$ and if $\overline{K}$ is compact in $L^2(\mathbb{R}^N)^{N+1}$,

$$K_1 = \Big\{\lambda^{N/2} \vec{v}(\lambda x) : \vec{v} \in K, c_0 \le \lambda \le c_1\Big\}$$

also has $\overline{K}_1$ is compact. Let now

$$\tilde{v}(x,t) = (1-t)^{N/2} \left(\nabla u((1-t)x,t), \partial_t u((1-t)x,t)\right),$$

so that $\tilde{v}(x,t) = \vec{v}(x + x(t), t)$. Since $\operatorname{supp} \vec{v}(\cdot, t) \subset \{x : |x - x(t)| \le 1\}$ and $E > 0$, the fact that $\{\vec{v}(\cdot, t)\}$ is compact implies that $|x(t)| \le M$. But if $K_2 = \{\vec{v}(x+x_0, t) : |x_0| \le M\}$, then $\overline{K}_2$ is compact, giving the Proposition. □

At this point we introduce a new idea, inspired by the works of Giga–Kohn [12] in the parabolic case and Merle–Zaag [36] in the hyperbolic case, who studied the equations $(\partial_t^2 - \triangle)u - |u|^{p-1}u = 0$, for $1 < p < \frac{4}{N-1} + 1$, in the radial case. In our case, $p = \frac{4}{N-2} + 1 > \frac{4}{N-1} + 1$. We thus introduce self-similar variables. Thus, we set $y = x/1-t$, $s = \log 1/1-t$ and define

$$w(y,s;0) = (1-t)^{N-2/2} u(x,t) = e^{-s(N-2)/2} u(e^{-s}y, 1-e^{-s}),$$

which is defined for $0 \le s < \infty$ with $\operatorname{supp} w(\cdot, s; 0) \subset \{|y| \le 1\}$. We will also consider, for $\delta > 0$, $u_\delta(x,t) = u(x, t+\delta)$ which also solves (7) and its corresponding w, which we will denote by $w(y,s;\delta)$. Thus, we set $y = x/1+\delta-t$, $s = \log 1/1+\delta-t$ and

$$w(y,s;\delta) = (1+\delta-t)^{N-2/2} u(x,t) = e^{-s(N-2)/2} u(e^{-s}y, 1+\delta-e^{-s}).$$

$w(y,s;\delta)$ is defined for $0 \le s < -\log\delta$ and we have

$$\operatorname{supp} w(\cdot, s; \delta) \subset \left\{|y| \le \frac{e^{-s} - \delta}{e^{-s}} = \frac{1-t}{1+\delta-t} \le 1 - \delta\right\}.$$

The w solve, where they are defined, the equation

$$\begin{aligned}\partial_s^2 w &= \frac{1}{\rho}\text{div }(\rho\nabla w - \rho(y\cdot\nabla w)y) - \frac{N(N-2)}{4}w + \\ &+ |w|^{4/N-2}w - 2y\cdot\nabla\partial_s w - (N-1)\partial_s w,\end{aligned}$$

where $\rho(y) = (1-|y|^2)^{-1/2}$.

Note that the elliptic part of this operator degenerates. In fact,

$$\frac{1}{\rho}\text{div }(\rho\nabla w - \rho(y\cdot\nabla w)y) = \frac{1}{\rho}\text{div }(\rho(I - y\otimes y)\nabla w)\,,$$

which is elliptic with smooth coefficients for $|y|<1$, but degenerates at $|y|=1$.

Here are some straightforward bounds on $w(\cdot;\delta)$ $(\delta>0)$: $w\in H_0^1(B_1)$ with

$$\int_{B_1} |\nabla w|^2 + |\partial_s w|^2 + |w|^{2^*} \le C.$$

Moreover, by Hardy's inequality for $H_0^1(B_1)$ functions, ([6])

$$\int_{B_1} \frac{|w(y)|^2}{(1-|y|^2)^2} \le C.$$

These bounds are uniform in $\delta>0$, $0<s<-\log\delta$. Next, following [36],we introduce an energy, which will provide us with a Lyapunov functional for w.

$$\begin{aligned}\tilde{E}(w(s;\delta)) &= \int_{B_1}\frac{1}{2}\left\{(\partial_s w)^2 + |\nabla w|^2 - (y\cdot\nabla w)^2\right\}\frac{dy}{(1-|y|^2)^{1/2}} + \\ &+ \int_{B_1}\left\{\frac{N(N-2)}{8}\,w^2 - \frac{N-2}{2N}|w|^{2^*}\right\}\frac{dy}{(1-|y|^2)^{1/2}}.\end{aligned}$$

Note that this is finite for $\delta>0$. We have:

Lemma 10. *For $\delta>0$, $0<s_1<s_2<\log 1/\delta$,*

i) $\tilde{E}(w(s_2)) - \tilde{E}(w(s_1)) = \displaystyle\int_{s_1}^{s_2}\int_{B_1}\frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}}dsdy$, *so that $\tilde{E}$ is increasing*

ii) $$\frac{1}{2}\int_{B_1}\left[(\partial_s w)\cdot w - \frac{1+N}{2}w^2\right]\frac{dy}{(1-|y|^2)^{1/2}}\Bigg|_{s_1}^{s_2} =$$

$$= -\int_{s_1}^{s_2}\tilde{E}(w(s))ds + \frac{1}{N}\int_{s_1}^{s_2}\int_{B_1}\frac{|w|^{2^*}}{(1-|y|^2)^{1/2}}dsdy +$$

$$+ \int_{s_1}^{s_2}\int_{B_1}\left\{(\partial_s w)^2 + \partial_s w y\cdot\nabla w + \frac{\partial_s w w|y|^2}{(1-|y|^2)}\right\}\frac{dy}{(1-|y|^2)^{1/2}}.$$

iii) $\lim_{s\to\log 1/\delta} \tilde{E}(w(s)) = E((u_0,u_1)) = E$, *so that, by i),* $\tilde{E}(w(s)) \leq E$ *for* $0 \leq s < \log 1/\delta$.

The proof is computational. See [23]. Our first improvement over this is:

Lemma 11. $\displaystyle\int_0^1 \int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)} dyds \leq C \log 1/\delta.$

Proof. Notice that

$$
\begin{aligned}
-2\int \frac{(\partial_s w)^2}{(1-|y|^2)} &= \frac{d}{ds}\left\{\int \left[\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}\left(|\nabla w|^2 - (y\cdot\nabla w)^2\right) + \right.\right.\\
&+ \left.\frac{(N-2)N}{8}w^2 - \frac{N-2}{2N}|w|^{2^*}\right]\left[-\log(1-|y|^2)\right] dy + \\
&+ \int \left[\log(1-|y|^2)+2\right] y\cdot\nabla w \partial_s w - \log(1-|y|^2)(\partial_s w)^2 - \\
&- 2\int (\partial_s w)^2.
\end{aligned}
$$

We next integrate in s, between 0 and 1 and drop the next to last term by sign. The proof is finished by using Cauchy–Schwartz and the support property of $w(\cdot;\delta)$. $\square$

Corollary 4. *a)* $\displaystyle\int_0^1 \int_{B_1} \frac{|w|^{2^*}}{(1-|y|^2)^{1/2}} dyds \leq C(\log 1/\delta)^{1/2}$

b) $\tilde{E}(w(1)) \geq -C(\log 1/\delta)^{1/2}$

Proof. a) follows from ii), iii) above, Cauchy–Schwartz and the previous Lemma. Note that we obtain the power $1/2$ on the right hand side by Cauchy–Schwartz. b) follows from i) and the fact that

$$\int_0^1 \tilde{E}(w(s))ds \geq -C(\log 1/\delta)^{1/2},$$

which is a consequence of the definition of $\tilde{E}$ and a). $\square$

Our next improvement is:

Lemma 12. $\displaystyle\int_1^{\log 1/\delta} \int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}} \leq C(\log 1/\delta)^{1/2}.$

Proof. Use i), iii) and the bound b) in the Corollary. $\square$

Corollary 5. *There exists* $\overline{s}_\delta \in \big(1, (\log 1/\delta)^{3/4}\big)$, *such that*

$$\int_{\overline{s}_\delta}^{\overline{s}_\delta+(\log 1/\delta)^{1/8}} \int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}} \leq \frac{C}{(\log 1/\delta)^{1/8}}.$$

Proof. Split $\big(1, (\log 1/\delta)^{3/4}\big)$ into disjoint intervals of length $(\log 1/\delta)^{1/8}$. Their number is $(\log 1/\delta)^{5/8}$ and $\frac{5}{8} - \frac{1}{8} = \frac{1}{2}$. $\square$

Note that in the Corollary, the length of the s interval tends to infinity, while the bound goes to zero. It is easy to see that if $\overline{s}_\delta \in \big(1, (\log 1/\delta)^{3/4}\big)$, and $\overline{s}_\delta = -\log(1+\delta-\overline{t}_\delta)$, then

$$\left| \frac{1-\overline{t}_\delta}{1+\delta-\overline{t}_\delta} - 1 \right| \leq C\delta^{1/4},$$

which goes to 0 with δ. From this and the compactness of $\overline{K}$, we can find $\delta_j \to 0$, so that $w(y, \overline{s}_{\delta_j} + s; \delta_j)$ converges, for $s \in [0, S]$ to $w^*(y, s)$ in $C([0, S]; \dot{H}_0^1 \times L^2)$, and w^* solves our self-similar equation in $B_1 \times [0, S]$. The previous Corollary shows that w^* must be independent of s. Also, the fact that $E > 0$ and our coercivity estimates show that $w^* \not\equiv 0$. (See [23] for the details). Thus, $w^* \in H_0^1(B_1)$ solves the (degenerate) elliptic equation

$$\frac{1}{\rho}\text{div } (\rho \nabla w^* - \rho(y \cdot \nabla w^*)y) - \frac{N(N-2)}{4} w^* + |w^*|^{4/N-2} w^* = 0,$$

$$\rho(y) = (1-|y|^2)^{-1/2}.$$

We next point out that w^* satisfies the additional (crucial) estimates:

$$\int_{B_1} \frac{|w^*|^{2^*}}{(1-|y|^2)^{1/2}} + \int_{B_1} \frac{\big[|\nabla w^*|^2 - (y \cdot \nabla w^*)^2\big]}{(1-|y|^2)^{1/2}} < \infty.$$

Indeed, for the first estimate it suffices to show that, uniformly in j large, we have

$$\int_{\overline{s}_{\delta_j}}^{\overline{s}_{\delta_j}+\delta} \int_{B_1} \frac{|w(y,s;\delta_j)|^{2^*}}{(1-|y|^2)^{1/2}} dyds \leq C,$$

which follows from ii) above, together with the choice of $\overline{s}_{\delta_j}$, by the Corollary, Cauchy–Schwartz and iii). The proof of the second estimate follows from the first one, iii) and the formula for $\tilde{E}$.

The conclusion of the proof is obtained by showing that a w^* in $H_0^1(B_1)$, solving the degenerate elliptic equation with the additional bounds above, must be zero. This will follow from a unique continuation argument. Recall that, for $|y| \leq 1-\eta_0$, $\eta_0 > 0$, the linear operator is uniformly eliptic, with smooth coefficients and that the non-linearity is critical. An argument of Trudinger's [51] shows that

w^* is bounded on $\{|y| \le 1 - \eta_0\}$ for each $\eta_0 > 0$. Thus, if we show that $w^* \equiv 0$ near $|y| = 1$, the standard Carleman unique continuation principle [19] will show $w^* \equiv 0$.

Near $|y| = 1$, our equation in modeled by (in variables $z \in \mathbb{R}^{N-1}$, $r \in \mathbb{R}$, $r > 0$, near $r = 0$)

$$r^{1/2}\partial_r(r^{1/2}\partial_r w^*) + \triangle_z w^* + cw^* + |w^*|^{4/N-2}w^* = 0.$$

Our information on w^* translates into $w^* \in H_0^1((0,1] \times (|z| < 1))$ and our crucial additional estimates are:

$$\int_0^1 \int_{|z|<1} |w^*(r,z)|^{2^*} \frac{dr}{r^{1/2}} dz + \int_0^1 \int_{|z|<1} |\nabla_z w^*(r,z)|^2 \frac{dr}{r^{1/2}} dz < \infty.$$

To conclude, we take advantage of the degeneracy of the equation. We "desingularize" the problem by letting $r = a^2$, setting $v(a,z) = w^*(a^2, z)$, so that $\partial_a v(a,z) = 2r^{1/2}\partial_r w^*(r,z)$. Our equation becomes:

$$\partial_a^2 v + \triangle_z v + cv + |v|^{4/N-2}v = 0, \quad 0 < a < 1, |z| < 1, \quad v|_{a=0} = 0,$$

and our bounds give:

$$\int_0^1 \int_{|z|<1} |\nabla_z v(a,z)|^2 da dz = \int_0^1 \int_{|z|<1} |\nabla_z w^*(r,z)|^2 \frac{dr}{r^{1/2}} dz < \infty,$$

$$\int_0^1 \int_{|z|<1} |\partial_a v(a,z)|^2 \frac{da}{a} dz = \int_0^1 \int_{|z|<1} |\partial_r w^*(r,z)|^2 dr dz < \infty.$$

Thus, $v \in H_0^1((0,1] \times B_1)$, but in addition $\partial_a v(a,z)|_{a=0} \equiv 0$. We then extend v by 0 to $a < 0$ and see that the extension is an H^1 solution to the same equation. By Trudinger's argument, it is bounded. But since it vanishes for $a < 0$, by Carleman's unique continuation theorem, $v \equiv 0$. Hence, $w^* \equiv 0$, giving our contradiction. $\square$

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